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# The construction of a Poisson structure out of a symmetry and a conservation law of a dynamical system

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**Abstract.** A method to construct Hamiltonian theories for systems of both ordinary and partial differential equations is presented. The knowledge of a Lagrangian is not at all necessary to achieve the result. The only ingredients required for the construction are one solution of the symmetry (perturbation) equation and one constant of motion of the original system. It turns out that the Poisson bracket structure for the dynamical variables is far from becoming uniquely determined by the differential equations of motion. Examples in classical mechanics as well as in field theory are presented.

Hamiltonian methods are widely used in connection with problems in classical, quantum and statistical mechanics, fluid dynamics, optics, solid state, molecular, atomic, nuclear, particle and plasma physics, in both classical and quantum field theoretical systems. Quantization schemes as well as group theoretical symmetry methods are examples of subject matters in which Hamiltonian structures are useful. Hamiltonian theories are usually constructed, starting from the knowledge of a Lagrangian, by well established methods for both the cases of regular and singular Lagrangians [1–4]. For different reasons, one may try to quantize or to construct Hamiltonian structures for classical systems of differential equations without recourse to a Lagrangian [5–7]. Several authors have been successful in creating Hamiltonian theories from scratch for different examples, mostly in fluid dynamics (an excellent review is presented in [8]) and in field theory [9], but no general method seems to exist for constructing a Hamiltonian structure starting from the equations of motion only, without using at all either the explicit form or the existence of a Lagrangian formulation for the system at hand.

The purpose of this paper is to present a general technique to construct Hamiltonian theories starting from the equations of motion, using one symmetry transformation and one constant of motion, without recourse to the Lagrangian of the system of equations, which may even fail to exist. This method has even allowed us to find a Hamiltonian description for the heat equation [10] which has been believed to be non-Hamiltonian for some time [11]. A completely different approach has been used to construct a quantum model for a non-Lagrangian cosmological model in [12], and infinitely many Hamiltonian structures of the spinning top in [13].

Let us start by stating what it is usually meant by a Hamiltonian theory. Consider an autonomous first-order differential system,

$$\frac{\mathrm{d}x^a}{\mathrm{d}t} = f^a(x^b) \qquad a, b = 1, \ldots, N. \tag{1}$$

 

Of course, a differential system of any order can be easily cast in first-order form by defining extra variables in the standard textbook fashion. A Hamiltonian structure for (1) is defined in terms of an antisymmetric Poisson matrix $\boldsymbol{J}^{ab}(x^c)$ and Hamiltonian $H(x^c)$ which satisfy

$$\boldsymbol{J}^{ab} = -\boldsymbol{J}^{ba} \qquad a, b, c, \ldots = 1, \ldots, N \tag{2}$$

$$\boldsymbol{J}^{ab}{}_{,d}\boldsymbol{J}^{dc} + \boldsymbol{J}^{bc}{}_{,d}\boldsymbol{J}^{da} + \boldsymbol{J}^{ca}{}_{,d}\boldsymbol{J}^{db} \equiv 0 \tag{3}$$

and

$$\boldsymbol{J}^{ab}\frac{\partial H}{\partial x^b} = f^a. \tag{4}$$

The Poisson bracket between any two dynamical variables $A(x^a)$ and $B(x^b)$ is defined by

$$[A, B] = \frac{\partial A}{\partial x^a}\boldsymbol{J}^{ab}\frac{\partial B}{\partial x^b} \tag{5}$$

and satisfies all the usual algebraic and differential properties as can be easily inferred from the antisymmetry condition (2), the Jacobi identity (3), and the definition (5).

If, in addition,

$$\det \boldsymbol{J}^{ab} \neq 0 \tag{6}$$

is required, then the Poisson matrix is regular. It is important to remark that, sometimes, condition (6) cannot be met. If $N$ is odd, as happens for Euler's equations, (6) is never satisfied because of (2) as can be seen for instance in [13] and in one of the examples below. In Dirac theory, condition (6) for Dirac brackets is not satisfied because there are dynamical entities, called second class constraints, which have vanishing Dirac brackets with any variable [3]. In fluid dynamics, the functions which have vanishing Poisson brackets with any dynamical variable are called 'Casimir functions', inspired on the well known group theoretical terminology for operators which commute with any element of the group [8, 14]. Therefore, it is convenient to adopt a flexible attitude regarding condition (6), and take conditions (2), (3), and (4) as defining a Hamiltonian theory. Of course, the usual textbook Hamiltonian structures satisfy all of them.

It is straightforward to prove that conditions (2) and (4), imply that $H(x^a)$ is a time-independent constant of the motion of system (1) defined by the condition

$$\frac{\partial H}{\partial x^a} f^a = \mathcal{L}_f H = 0 \tag{7}$$

which can be equivalently stated by saying that the Lie derivative of $H$ along $f$ vanishes. A brief account on Lie derivatives may be found in [15].

Let us now derive the symmetry (perturbation) equation of (1). (For a detailed discussion, see [16].) Consider the transformation

$$\tilde{x}^a = x^a + \epsilon\eta^a(x^b, t) \tag{8}$$

where $\epsilon\eta^a(x^b, t)$ is a small perturbation which maps solutions of (1) into solutions of the same equation, up to first order in $\epsilon$. The equation that the perturbation vector $\eta$ satisfies is

$$\partial_t\eta^a + \eta^a{}_{,b}f^b - f^a{}_{,b}\eta^b = 0 \tag{9}$$

or

$$(\partial_t + \mathcal{L}_f)\eta^a = 0. \tag{10}$$

It is not difficult to prove that $K$, the deformation of $H$ along $\eta$, defined by

$$K \equiv \frac{\partial H}{\partial x^a}\eta^a = \mathcal{L}_\eta H \tag{11}$$

is also a constant of motion for the same system, if $\eta$ satisfies equation (10). By the same token, a new symmetry transformation $\bar{\eta}$ which satisfies equation (10) can be constructed using a symmetry transformation $\eta$ and a constant of motion $K$ by

$$\bar{\eta}^a = \frac{\eta^a}{K}. \tag{12}$$

A detailed account of these results may be found in [17].

Let us now compute the Lie derivative of $\boldsymbol{J}^{ab}$ along $f$

$$\mathcal{L}_f \boldsymbol{J}^{ab} = \boldsymbol{J}^{ab}{}_{,c} f^c - \boldsymbol{J}^{ac} f^b{}_{,c} - \boldsymbol{J}^{cb} f^a{}_{,c}. \tag{13}$$

It is a straightforward exercise to prove that

$$\mathcal{L}_f \boldsymbol{J}^{ab} = 0 \tag{14}$$

using equations (2)–(4). Note that equation (6) is not needed in the proof. Therefore, a Poisson matrix must have vanishing Lie derivative along $f$. Nevertheless, this condition is not sufficient to fulfil simultaneously the requirements (2)–(4). To construct a Poisson matrix $\boldsymbol{J}^{ab}$, let us start by considering an antisymmetric matrix according to the following *ansatz*

$$\boldsymbol{J}^{ab} = f^a \eta^b - f^b \eta^a \tag{15}$$

where $\eta$ satisfies (9) and has been normalized using (12) in such a way that $\boldsymbol{J}^{ab}$ fulfil (4) identically. Of course, condition (2) is trivially met. The Jacobi identity (3) imposes the following condition

$$\boldsymbol{J}^{bc} \mathcal{L}_f \eta^a + \boldsymbol{J}^{ca} \mathcal{L}_f \eta^b + \boldsymbol{J}^{ab} \mathcal{L}_f \eta^c = 0 \tag{16}$$

which is satisfied by a particular, time-independent symmetry vector $\eta_0$ which solves (10) defined by

$$\partial_t \eta_0^a = -\mathcal{L}_f \eta_0^a = 0. \tag{17}$$

A more interesting solution $\eta_1$ is given by the condition

$$\partial_t \eta_1^a = -\mathcal{L}_f \eta_1^a = \lambda f^a \tag{18}$$

which will be most useful in many instances. Note that both solutions produce Poisson matrices with vanishing Lie derivatives along $f$.

We have thus constructed a Hamiltonian structure for (1) based on knowledge of just one symmetry vector (either $\eta_0$ or $\eta_1$) and only one constant of motion, $H$, of the system under consideration (assuming a non-vanishing $K$, which can be easily achieved as will be seen in the examples). A few comments seem in order. First, a solution similar to the one described by (18), with the Lie derivative of the symmetry vector along $f$ proportional to the symmetry vector itself, although it satisfies the Jacobi identity, is incompatible with (4). Second, the rank of the Poisson matrix just derived is two. Therefore, it will be, in most cases, singular. A procedure to enlarge its rank will be described below. Third, it is obvious that the method we have presented will, in general, yield a Hamiltonian structure written in terms of non-canonical coordinates. Nevertheless, that is the most we can hope for in the case of non-Lagrangian systems (which are always described by non-commutative geometry). This problem is dealt with in some detail in [7]. Fourth, even though this procedure differs from the usual one for the case of Lagrangian systems, it may sometimes reproduce the well known results in terms of canonical coordinates, as is shown in one of the examples below. Let us now consider some examples. The systems may be completely described by the evolution vector $f$, or the equations of motion written in its first-order version. The Hamiltonian structure may be completely determined by one constant of motion, the Hamiltonian $H$, and one symmetry vector $\eta_0$ or $\eta_1$. Sometimes, we will need to make use of the normalization given in (12).

*Example 1. One-dimensional monomial force*. This example is defined by the equations of motion

$$f^1 = x^2 \qquad f^2 = -c(n+1)(x^1)^n \tag{19}$$

while a Hamiltonian

$$H = \frac{(x^2)^2}{2} + c(x^1)^{n+1} \tag{20}$$

and one symmetry transformation are given by

$$\eta^1 = x^1 + \frac{n-1}{2} t f^1 \qquad \eta^2 = \frac{n+1}{2} x^2 + \frac{n-1}{2} t f^2. \tag{21}$$

This is, of course, a very trivial example, which, nonetheless, shows how the scheme presented here can reproduce the usual results. In this case, the Poisson matrix is regular. The harmonic oscillator and the free particle are special cases in this example. Note that this treatment can be extended to any number of dimensions provided the force is a homogeneous function of degree $n$ in the coordinates.

*Example 2. Euler's top*. Consider the equations of motion of Euler's top

$$\frac{\mathrm{d}L^i}{\mathrm{d}t} = -\epsilon^{ijk}\Omega_j L_k \equiv f^i \qquad i, j, k = 1, 2, 3 \tag{22}$$

with

$$\Omega_j = \frac{L_j}{I_j} \tag{23}$$

where $L_i = L^i$ and $\Omega_i = \Omega^i$ are the components of the angular momentum vector and the angular velocity vector in the $i$th principal direction, respectively, and the $I_i$ is the eigenvalue of the tensor of inertia of an asymmetrical top along with the $i$th principal axis, as usual.

Let us now look for symmetries of the equations of motion. With this purpose in mind, multiply the angular momentum by some constant factor $\lambda$. This operation introduces a $\lambda^2$ factor in the right-hand side of the equation of motion (22). The same result is achieved in the left-hand side of the equation if, in addition, time is multiplied by the inverse factor $\lambda^{-1}$. These operations performed simultaneously constitute a finite symmetry transformation for (22). One can deal with an infinitesimal version of it by considering $\lambda = 1 + \zeta$ infinitesimally close to one, to get the transformation

$$\delta L^i = \zeta L^i \qquad \delta t = -\zeta t. \tag{24}$$

Note that a transformation such as (24) may equivalently be written as (see, for instance [16])

$$\eta^i = \zeta(L^i + t\epsilon^{ijk}\Omega_j L_k) \tag{25}$$

leaving time invariant. It is now a straightforward matter to check that the transformation defined by (25) is, in fact, a symmetry transformation for (22) because it satisfies (9). Note that $\eta^i$ also satisfies (18), and consequently may, in principle, be used to define a Poisson matrix according to (15).

It is well known that $C_1$ and $C_2$ given by

$$C_1 = (L^1)^2 + (L^2)^2 + (L^3)^2 \tag{26}$$

and

$$C_2 = \frac{(L^1)^2}{2I_1} + \frac{(L^2)^2}{2I_2} + \frac{(L^3)^2}{2I_3} \tag{27}$$

are constants of motion for the dynamics generated by (22). We have already seen that the Hamiltonian for any system must be a constant of motion. Therefore, $C_1$ and $C_2$ are, in principle, possible Hamiltonians for the top.

The deformations of $C_1$ and $C_2$ along $\eta^i$ do not vanish, in fact,

$$K_1 \equiv \frac{\partial C_1}{\partial L^i}\eta^i = 2C_1 \tag{28}$$

and

$$K_2 \equiv \frac{\partial C_2}{\partial L^i}\eta^i = 2C_2. \tag{29}$$

We have thus found two inequivalent Hamiltonian formulations for the top, defined by the Poisson matrices $\boldsymbol{J}_1^{ij}$ and $\boldsymbol{J}_2^{ij}$ and the Hamiltonians $H_1$ and $H_2$ given by

$$\boldsymbol{J}_1^{ij} = \frac{1}{K_1}(f^i\eta^j - f^j\eta^i) \tag{30}$$

$$H_1 = C_1 \tag{31}$$

$$\boldsymbol{J}_2^{ij} = \frac{1}{K_2}(f^i\eta^j - f^j\eta^i) \tag{32}$$

and

$$H_2 = C_2. \tag{33}$$

Note that the choice of a Hamiltonian $H$ as an arbitrary function $H = H(C_1, C_2)$ is also possible provided the proper normalization factor $K$ is used in the Poisson matrix $\boldsymbol{J}^{ij}$. In this way, we have constructed infinitely many Hamiltonian structures for Euler's top.

*Example 3. Radial forces.* This example considers non-potential radial forces defined by

$$f^i = x^{i+3} \qquad f^{i+3} = Fx^i \qquad F = F(\boldsymbol{r}^2, \dot{\boldsymbol{r}}^2, \boldsymbol{r}\cdot\dot{\boldsymbol{r}}) \qquad i = 1, 2, 3. \tag{34}$$

To construct a Hamiltonian structure, the Hamiltonian may be chosen to be the third component of the (conserved) angular momentum vector, while the symmetry transformation is, for instance, a rotation around the first axis. This example clearly shows the ambiguity which exists to choose both the Hamiltonian and the Poisson matrix.

*Example 4. Korteweg–de Vries equation.* The equation of motion is

$$u_t = -uu_x - u_{xxx} \equiv f. \tag{35}$$

We are now going to construct a symmetry transformation for it. Take any solution of equation (35) and define a new set of variables $u'$, $x'$, and $t'$ by multiplying the old variables $u$, $x$, $t$ by factors $\lambda^{-2}$, $\lambda$, $\lambda^3$, respectively. This operation simply produces an overall $\lambda^5$ factor in the equation, which means that the new set of variables solves the same equation which the old variables satisfy. We have thus constructed a finite symmetry transformation for the Korteweg–de Vries equation. The infinitesimal symmetry associated with it may be written taking $\lambda = 1 + \zeta$, infinitesimally close to one

$$\delta u = -2\zeta u \qquad \delta x = \zeta x \qquad \delta t = 3\zeta t \tag{36}$$

or, equivalently,

$$\eta = \zeta(-2u - xu_x + 3t(uu_x + u_{xxx})) \tag{37}$$

leaving $x$ and $t$ unchanged [16]. It is now a straightforward matter to prove that $\eta$ is an infinitesimal symmetry transformation for (35) because it satisfies the symmetry (perturbation) equation (9). We remark that $\eta$ satisfies condition (18) and therefore, it can be used to construct a Poisson matrix. One possible choice for the Hamiltonian density is $u^2$. The Hamiltonian $H$

$$H = \int u^2 \, \mathrm{d}x \tag{38}$$

is a constant of motion, i.e. its time derivative vanishes when the usual assumptions about the behaviour of the fields at spatial infinity are adopted. In fact, the time derivative of the Hamiltonian can be written as the integral of a total spatial divergence (a partial derivative with respect to $x$, in our case) when the equation of motion (35) is taken into account.

One gets that the deformation $K$ of $H$ along $\eta$ is non-vanishing. We can easily see that the functional derivative of $H$ in the $\eta$ direction is

$$K \equiv \int \frac{\delta H}{\delta u(x)} \eta(x) \, \mathrm{d}x = -3\zeta H \tag{39}$$

as it can be obtained by direct computation, or by taking advantage of the new variables defined by multiplying the old ones by powers of λ as we have already done above.

Therefore, one Poisson structure is given by

$$\boldsymbol{J}(x, y) = \frac{1}{K}(f(x)\eta(y) - f(y)\eta(x)). \tag{40}$$

The equation of motion can now be written in Hamiltonian form

$$u_t = [u, H] \tag{41}$$

where the field theoretical Poisson bracket is defined, as usual, in terms of functional derivatives by

$$[A, B] \equiv \int \frac{\delta A}{\delta u(x)} \boldsymbol{J}(x, y) \frac{\delta B}{\delta u(y)} \, \mathrm{d}x \, \mathrm{d}y. \tag{42}$$

Note that other Hamiltonian densities $H'$ can be used as well, in conjunction with the same symmetry vector $\eta$, provided the corresponding deformations $K'$ be used in the definition of the new Poisson matrix $\boldsymbol{J}'(x, y)$.

*Example 5. Nonlinear Schrödinger equations.* The equations of motion are

$$i\psi_t + \psi_{xx} + \psi^2\psi^* = 0 \tag{43}$$

and its complex conjugate. One possible non-standard Hamiltonian density is $\psi\psi^*$, and the symmetry vectors are

$$\eta = -\psi - x\psi_x + 2t(\psi_{xx} + \psi^2\psi^*) \tag{44}$$

and its complex conjugate. This Hamiltonian structure appears to be new.

Even though most of the examples presented here have time-dependent symmetry vectors $\eta^a$, the Poisson matrices constructed out of them through relation (15) are, in fact, time-independent. This is so because the time-dependent part of each symmetry vector is (in our examples) proportional to the vector $f^a$ and therefore, no time-dependence appears in $\boldsymbol{J}^{ab}$ due to the antisymmetric product in (15). Furthermore, there are other systems for which a Poisson structure may be constructed using time-independent symmetry vectors $\eta^a$, such as the one considered in the third example.

We have constructed Hamiltonian structures for several systems, starting from the equations of motion only. Some of these structures, have singular Poisson matrices. One way to increase the rank of the Poisson matrix, without altering any other of its properties, is the following. Assume we can find two new time-independent symmetry vectors $\eta_2$ and $\eta_3$ such that the Lie derivatives of the Hamiltonian $H$ along them vanish, i.e.

$$\frac{\partial H}{\partial x^a}\eta_2^a = \frac{\partial H}{\partial x^a}\eta_3^a = 0 \tag{45}$$

and that the Lie derivatives of $\eta_2$ along $\eta_3$, as well as those of $\eta_2$ and $\eta_3$ along $\eta_1$ (or $\eta_0$), vanish. Then, the new Poisson matrix $\boldsymbol{J}_1^{ab}$ defined by

$$\boldsymbol{J}_1^{ab} = \boldsymbol{J}^{ab} + \eta_2^a\eta_3^b - \eta_3^a\eta_2^b \tag{46}$$

satisfies all of the requirements which define a Poisson matrix (2), (4), and even the non-linear Jacobi identity (3), and its rank is equal to four. This procedure can be repeated at will, producing an increase of two units in the rank of the Poisson matrix each time that it is performed. (Note that this construction clearly shows that the Poisson bracket structure is not uniquely determined by the dynamics.) If, eventually, one gets a regular Poisson matrix, the method presented here may constitute an alternative to construct a Lagrangian description of the system (1), yielding a novel, symmetry based, approach to the classical inverse problem of the calculus of variations [18–22].

Note that the choice of the symmetry vector ($\eta_0$ or $\eta_1$) needed to define the Poisson matrix is determined solely by the requirement of getting a non-vanishing $K$, given $H$. We have used in the examples both time-dependent and time-independent symmetry vectors, to illustrate different possibilities. Sometimes, there may be several adequate choices of symmetry vectors for a given $H$, producing different Hamiltonian formulations for the same system. We remark that singular Poisson matrices are present in Dirac's construction of Hamiltonian structures, as we have already mentioned above. We are currently investigating the possibility of applying this method, which naturally leads to singular Poisson matrices, to deal with gauge and constrained systems, as an alternative to Dirac's method when no Lagrangian is available. We are also studying whether it is possible to obtain constants of the motion of the system at hand as Casimir functions of the singular Poisson matrix constructed here.

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